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APPLICATIONS OF DUALITY AND STOCHASTIC
DOMINANCE IN RELIABILITY THEORY

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ABSTRACT

In this paper we discuss a notion which is the dual of the failure rate and is defined as $f(t)/F(t)$. This concept has come up several times in the literature, but has never been investigated in detail. In this paper the name *survival rate* is used. We show that in many models of practical interest the survival rates of certain random variables have nice properties. This makes it possible to obtain bounds on the distribution functions of these random variables. Moreover, we show that a form of stochastic dominance based on the survival rates of the random variables has some interesting applications.

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APPLICATIONS OF DUALITY AND STOCHASTIC DOMINANCE
IN RELIABILITY THEORY

A well-known characteristic of the distribution of the lifetime of a component is its failure rate $r(t)$, where $r(t)$ is defined as $f(t)/(1-F(t))$. One can imagine $r(t)$ to be the probability of failure of a component during the interval $(t, t+x]$ given that the component is still alive at time t , and where the size of the interval, x , is very small. So

$$r(t) = \lim_{x \rightarrow 0} \frac{1}{x} \frac{F(t+x) - F(t)}{(1 - F(t))} = \frac{f(t)}{1 - F(t)}$$

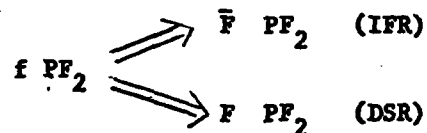
The following notion can be considered as the dual of the failure rate: The probability of failure of a component during the interval $[t-x, t)$ given that the component is not alive at time t and where again the size of the interval, x , is very small. As this is the conditional probability of the component surviving to time $t-x$, with x very small, we will use the term survival rate and will denote it by $s(t)$, i.e.

$$s(t) = \lim_{x \rightarrow 0} \frac{1}{x} \frac{F(t) - F(t-x)}{F(t)} = \frac{f(t)}{F(t)}$$

This concept has been briefly discussed by Barlow, Marshall and Proschan (1963, page 380) and by Kelson and Sumita (1980, page 25). However, in neither one of these papers it was given a name. Barlow, Marshall and Proschan (1963) considered distributions with $f(t)/F(t)$ decreasing in t , i.e. distributions with a *Decreasing Survival Rate* (DSR). They noted that the Decreasing Survival Rate (DSR) property, just like the Increasing Failure

Rate (IFR) property, is preserved under convolution. This was shown by observing that when the random variable X is IFR the random variable $-X$ is DSR.

In reliability theory distributions with monotone failure rates are very important. Increasing Failure Rate distributions have some well-known and useful properties. The most important property is probably the fact that IFR-ness is preserved under convolution. A second property is the fact that the lifetime of a k out of n system is IFR when it is built up of components with independent identical IFR distributions (see Barlow and Proshan (1975), page 107). It is well-known that F being IFR is equivalent to $\log(\bar{F}(x))$ being concave or \bar{F} being a Polya frequency function of order 2 (PF_2). Barlow et al. (1963) also observed that a distribution F is DSR if and only if $\log(F(x))$ is concave. The following relationships hold between IFR distributions, DSR distributions and distributions of which the densities are PF_2 :



In order to get some intuitive feeling for these classes of distributions, observe that a PF_2 density function implies that the density function is unimodal, while IFR-ness implies that there are no downward jumps in the density function and DSR-ness implies that there are no upward jumps in the density function. For examples of PF_2 densities, see problem 7 of page 79 of Barlow and Proshan (1975). The class of DSR distributions is

fairly large as it comprises the class of DFR distributions as well as the class of distributions of which the densities are PF_2 . At the other hand, one can show easily that the class of distributions with an *Increasing Survival Rate* is *empty*. Even random variables on a finite support cannot have an Increasing Survival Rate.

In reliability theory a distribution F is called Increasing Failure Rate on the Average (IFRA) if $\int_0^t r(x)dx/t$ is increasing in t . This is equivalent to $-(1/t) \log(1-F(t))$ being increasing in t . A dual of the IFRA concept can only be defined for distributions with an upper bound U . This dual property then implies that $\int_t^U s(x)dx/(U-t)$ decreases in t . It is clear that this concept cannot be very useful in practice as it is only defined for random variables on a finite support.

In this paper we will give some examples of stochastic models where the survival rates of specific random variables play an important role and give valuable additional information about the models. In Section 1 we give examples of popular models where the random variables of interest are DSR. In Section 2 we discuss a form of stochastic dominance between random variables based on their survival rates and give some examples where this type of dominance can occur. In Section 3 we present bounds on the distributions of DSR random variables based on the knowledge of a mean and a percentile. In the last Section we discuss how this notion can be used in the analysis of stochastic models.

1. Examples of Random Variables with a Decreasing Survival Rate (DSR).

In this section we discuss a number of stochastic models in which the survival rate of certain random variables play an important role. The

proofs are in some cases rather easy and will then be left to the reader.

(i) *The Lifetime of a Parallel System.* Let X_i , $i=1, \dots, n$, be a random variable with survival rate $s_i(t)$. Let $Y = \max(X_1, \dots, X_n)$ and let $s(t)$ denote the survival rate of Y . We present the following proposition without proof.

Proposition 1. The survival rate of the lifetime of a parallel system can be obtained through the summation of the survival rates of the individual components, i.e. $s(t) = \sum_{i=1}^n s_i(t)$ for all t .

Recall that the failure rate of the lifetime of a series system can be obtained through the summation of the failure rates of the individual components.

(ii) *The Lifetime of a k out of n System.* Let

$$x_i = \begin{cases} 1 & \text{if component } i \text{ is functioning} \\ 0 & \text{otherwise} \end{cases}$$

and let $\bar{x} = (x_1, \dots, x_n)$. Let the structure function ϕ of a system be defined as follows.

$$\phi(\bar{x}) = \begin{cases} 1 & \text{if the system is functioning under } \bar{x} \\ 0 & \text{otherwise} \end{cases}$$

If X_i , $i=1, \dots, n$, are assumed to be independent binary random variables with $P(X_i=1) = p_i = 1 - P(X_i=0)$ then we define the reliability function $h(\bar{p}) = P(\phi(\bar{X})=1) = E(\phi(\bar{X}))$. For any coherent structure ϕ (not necessarily k out of n) a dual structure ϕ_D can be defined, where $\phi_D(\bar{x}) = 1 - \phi(\bar{1}-\bar{x})$ (see Barlow

and Proshan (1975), page 12). Let $h_\phi(\bar{p})$ denote the reliability function of the primal system. It is a well-known fact that $\bar{F}(t) = h_\phi(\bar{F}_1(t), \dots, \bar{F}_n(t))$, where $\bar{F}_1(t) (= 1 - F_1(t))$ is the probability of component 1 reaching age t and $\bar{F}(t) (= 1 - F(t))$ is the probability of the system reaching age t . It can also be shown easily then that $F(t) = h_{\phi_D}(F_1(t), \dots, F_n(t))$. This has the following consequences: If for the primal system ϕ it is known that when the distributions F_i , $i=1, \dots, n$ are IFR, the failure rate of the system is increasing (and/or concave, convex, unimodal), then the distributions F_i being DSR implies that the survival rate of the lifetime of the dual system is decreasing (and/or concave, convex, unimodal). This leads us to the following proposition.

Proposition 2. The lifetime of a k out of n system, built up of components with DSR lifetimes that are i.i.d., is DSR.

Proof: This follows from the fact that IFR-ness is preserved under formation of a k out of n system with i.i.d. component lifetimes and the fact that the dual of a k out of n system is an $n-k+1$ out of n system. ■

(iii) *The Number of Components that are Down upon System Failure.* Consider a system where the lifetimes of the components are i.i.d. continuous random variables. Let N be the number of components that are off when the system goes off. El Newehi, Proshan and Sethuraman (1978) and Ross, Shashahani and Weiss (1980) have studied the properties of N . The random variable N is discrete and the failure rate is defined as

$$\lambda_k = \frac{P(N=k)}{P(N \geq k)} \quad k=1, 2, \dots$$

Ross et al. (1980) have shown that N is IFRA, i.e. $(1/k) \sum_{j=1}^k \lambda_j$ is nondecreasing in k . Moreover they showed that if the system has nonoverlapping

minimal cut sets, i.e. the minimal cut sets have no components in common, N is IFR, which implies that $P(N=k|N \geq k)$ is nondecreasing in k .

Proposition 3. When the system has nonoverlapping minimal path sets, i.e. the minimal path sets have no components in common, N is DSR, which implies that $P(N=k|N \leq k)$ is nonincreasing in k .

Proof: This was shown by Ross et al. (1980) through a duality argument. The reader should note however that there is a slight error in this paper; corollary 2 on page 364 should read *nonincreasing* instead of *nondecreasing*. ■

(iv) *The Poisson Shock Model.* A system is subject to shocks which occur according to a Poisson process with rate λ . The i -th shock causes a random amount of damage X_i , where the X_i , $i=1,2,\dots$, are i.i.d. with distribution F . The system fails when the total accumulated damage exceeds a given threshold x . Let Y denote the lifetime of the system and let H denote the distribution of Y . This model has been thoroughly analyzed by Esary, Marshall and Proschan (1973). They proved part (a) of the following proposition.

Proposition 4.

- (a) When the distribution F is DSR, the distribution H is IFR.
- (b) When the distribution F is IFR, the distribution H is DSR.

Proof: The first part is proven in Theorem 4.9 and Theorem 3.2. of Esary et al. (1973). The proof of the second part is similar with regard to the part corresponding to Theorem 4.9 and easier with regard to the part corresponding to Theorem 3.2. ■

The next example concerns some popular queueing models.

(v) *The GI|G|1 and GI|D|c queues.* Let S_i denote the service time of customer i , T_i the interarrival time between the i -th and the $(i+1)$ th arrival and D_i the delay of customer i (the time customer i spends in queue). It is well-known that for the GI|G|1 queue under the FIFO discipline the following recursive relationship holds:

$$D_{n+1} = \max(0, D_n + S_n - T_n).$$

This leads us to the following proposition.

Proposition 5. If S_i , $i=1,2,\dots$, is DSR and T_i , $i=1,2,\dots$, is IFR, D_i , $i=1,2,\dots$, is DSR.

Proof: This follows from the fact that $-T_i$, $i=1,2,\dots$, is DSR (so $D_n + S_n - T_n$ is DSR) and the fact that the maximum of two independent DSR random variables is again DSR. Clearly, the time in system of each customer is also DSR. ■

That this result can be extended to the GI|D|c system follows from the fact that under the FIFO discipline customer $c.i$, $i=1,2,\dots$ may be assigned to station c , customer $c.i + 1$, $i=0,1,2,\dots$ to station 1, customer $c.i + 2$, $i=0,1,2,\dots$ to station 2, etc. Each station may then be viewed as a GI|D|1 system.

The last three examples concern stochastic scheduling models.

(vi) *The Makespan of Jobs Subject to Series-Parallel Precedence Constraints.*

Consider a set of jobs subject to precedence constraints, where the precedence constraints have the form of a series-parallel digraph. For a discussion of these graphs see Lawler (1978). Lawler defines the class of

series-parallel graphs recursively as follows:

- (a) A single node with no arcs, e.g. $G = (\{1\}, \emptyset)$, is series-parallel.
- (b) If $G_1 = (N_1, A_1)$ and $G_2 = (N_2, A_2)$, where $N_1 \cap N_2 = \emptyset$, are series-parallel, then $G = G_1 \times G_2 = (N_1 \cup N_2, A_1 \cup A_2 \cup N_1 \times N_2)$ is also series-parallel; G is said to be a series composition of G_1 and G_2 .
- (c) If $G_1 = (N_1, A_1)$ and $G_2 = (N_2, A_2)$ where $N_1 \cap N_2 = \emptyset$ are series-parallel then $G = G_1 \cup G_2 = (N_1 \cup N_2, A_1 \cup A_2)$ is also series-parallel; G is said to be formed by the parallel composition of G_1 and G_2 .
- (d) Only those digraphs which can be obtained by a finite number of applications of rules a, b and c are series-parallel.

Proposition 6. When the processing times of the jobs are independent random variables with DSR distributions, and the capacity to process jobs is unlimited, the distribution of the makespan of the project is DSR.

Proof: This is true because any series composition of sets of jobs is equivalent to a convolution of independent random variables and any parallel composition results in a random processing time that is the maximum of two independent random variables. DSR-ness is preserved under both operations. ■

(vii) *The Two Machine Flow Shop with Zero Intermediate Storage.* In this model there are n jobs which have to be put at time $t=0$ in a sequence j_1, j_2, \dots, j_n . Job j_1 is processed first on the first machine. After finishing its processing there it goes to the second machine while job j_2 starts its processing on the first machine, etc. When job j_i has finished its processing on the first machine before job j_{i-1} has finished its processing on the second machine, blocking may occur, which implies that job j_i has to remain on the first machine, preventing subsequent jobs of being processed

on the first machine.

Proposition 7. When the processing times of all the jobs on the two machines are DSR, the makespan, i.e. the time to complete all the jobs, is DSR.

Proof: The time job j_i , $i=2, \dots, n$, occupies the first machine is the maximum of the processing time of job j_i on machine 1 and the processing time of job j_{i-1} on machine 2. The makespan is the convolution of the times that job j_i , $i=1, \dots, n$, occupy machine 1 and the time that job j_n occupies machine 2. ■

(viii) *The Two Machine Flow Shop with Infinite Intermediate Storage.* In this model, because of the infinite intermediate storage between the machines, blocking does not occur. When job j_i completes its processing on machine 1 before job j_{i-1} completes its processing on machine 2, job j_i is stored in between the two machines and machine 1 starts processing job j_{i+1} . We present the following proposition without proof.

Proposition 8. When the processing times of the jobs on machine 1 are deterministic, not necessarily identical, and the processing times on machine 2 are DSR, the makespan is DSR.

2. Stochastic Dominance Based on Failure Rates and Survival Rates

Lehman (1955) introduced an ordering between random variables which he called Monotone Likelihood Ratio Ordering (MLRO). Two continuous random variables X_1 and X_2 are said to be increasing monotone likelihood ratio ordered if for $s < t$

$$\frac{f_2(t)}{f_1(t)} \geq \frac{f_2(s)}{f_1(s)}$$

Pinedo and Ross (1980) defined two random variables to be increasing failure rate ordered if for all $s < t$

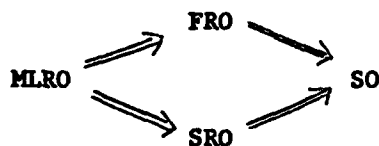
$$\frac{\bar{F}_1(t)}{\bar{F}_2(t)} \geq \frac{\bar{F}_1(s)}{\bar{F}_2(s)}$$

They showed that this ordering was equivalent to $r(t) \leq r_2(t)$ for all t , where $r_1(t)$ ($r_2(t)$) is the failure rate of X_1 (X_2). Moreover Pinedo and Ross (1980) showed that Monotone Likelihood Ratio Ordered (MLRO) \Rightarrow Failure Rate Ordered (FRO) \Rightarrow Stochastic Ordered (SO).

Keilson and Sumita (1980) introduced an ordering between X_1 and X_2 which we will call Survival Rate Ordering (SRO). Two continuous random variables X_1 and X_2 are said to be increasing survival rate ordered if for $s < t$

$$\frac{F_1(t)}{F_2(t)} \leq \frac{F_1(s)}{F_2(s)}$$

Using the identity $F(x) = \exp(-\int_x^\infty s(t)dt)$ one can easily show that this ordering is equivalent to $s_1(t) \leq s_2(t)$ for all t , where $s_1(t)$ ($s_2(t)$) is the survival rate of X_1 (X_2). It can also be established that Monotone Likelihood Ratio Ordered (MLRO) \Rightarrow Survival Rate Ordered (SRO) \Rightarrow Stochastic Ordered (SO), which results in the following chains of implications



Two random variables that are at the same time both FRO and SRO are not necessarily MLRO.

In the remaining part of this section Stochastic Ordering is denoted by \leq_{so} , Survival Rate Ordering by \leq_{sr} , Failure Rate Ordering by \leq_f and Monotone Likelihood Ratio Ordering by \leq_m . Two examples of applications of these orderings are now given.

(i) *Orderings Between Lifetimes of k Out of n Systems.* Consider a k out of n system with components that are independent and identically distributed with distribution F_1 . Consider a second k out of n system with components that are independent and identically distributed with distribution F_2 .

Proposition 9. If F_1 and F_2 are Failure (Survival) Rate Ordered, the lifetimes of the two systems are Failure (Survival) Rate Ordered in the same sense as F_1 and F_2 .

Proof: We will prove this proposition only for FRO random variables as the proof for SRO random variables is identical. Let $\lambda(t)$ denote the failure rate of a system of identical components with failure rates $r(t)$. It is well-known that

$$\lambda(t) = r(t) \frac{p \cdot h'(p)}{h(p)} \Big|_p = \bar{F}(t)$$

where $h(p) = h(p_1, \dots, p_n)$ is the reliability function of the system as defined in example (ii) of Section 1 (see Barlow and Proschan (1975), page 109, problem 6). It can be shown easily that $p \cdot h'(p)/h(p)$ decreases in p . The proposition then follows for the FRO case. The proof of the proposition for the SRO case is similar. ■

(ii) *Optimal System Assembly.* Optimal system assembly has been discussed several times in the literature, see Derman, Lieberman and Ross (1972, 1974). The models discussed here are slightly different. Two systems are considered: A series system and a parallel system, both with n component locations. In each location the wear on the component installed is different. Let w_j denote the wearout rate of location j , i.e. w_j represents the expected amount of wear sustained by a component in location j per unit time. There are n non-identical components available. The random variable X_i , $i=1, \dots, n$, with distribution F_i represents the total amount of wear a component can sustain before failure. Suppose we wish to minimize the probability of the system still being alive at time t . Let Y_j be the random amount of wear a component has to undergo in location j during a period t . So Y_j is a random variable with mean $w_j \cdot t$. To analyze this example further we need the following result from Brown and Solomon (1973).

Proposition 10: Assume the random variables X_1 and X_2 to be MLRO with $E(X_1) > E(X_2)$ and $g(x, y)$ is a real valued function satisfying $g(y, x) > g(x, y)$ for $y > x$. Then

$$g(X_2, X_1) \geq_{so} g(X_1, X_2)$$

Proof: See Lemma 1 in Brown and Solomon (1973).

Consider now a series system with n component locations. There are n components available to install in these n locations and we wish to maximize the probability of the system reaching age t .

Proposition 11: (a) If $Y_1 \leq_m Y_2 \leq_m \dots \leq_m Y_n$ and $X_1 \leq_f X_2 \leq_f \dots \leq_f X_n$ the probability of the series system reaching age t is maximized when assigning component i to location i , for $i=1, \dots, n$.

(b) If $Y_1 \leq_{sr} Y_2 \leq_{sr} \dots \leq_{sr} Y_n$ and $X_1 \leq_m X_2 \leq_m \dots \leq_m X_n$ the probability of the series system reaching age t is maximized when assigning component i to location i , for $i=1, \dots, n$.

Proof: The proposition can be shown easily through an adjacent pairwise interchange argument and proposition 10. ■

Consider a parallel system with n component locations. Again there are n components available to install in these n locations and again we wish to maximize the probability of the system reaching age t .

Proposition 12: (a) If $Y_1 \leq_m Y_2 \leq_m \dots \leq_m Y_n$ and $X_1 \geq_{sr} X_2 \geq_{sr} \dots \geq_{sr} X_n$ the probability of the parallel system reaching age t is maximized when assigning component i to location i , for $i=1, \dots, n$.

(b) If $Y_1 \geq_f Y_2 \geq_f \dots \geq_f Y_n$ and $X_1 \leq_m X_2 \leq_m \dots \leq_m X_n$ the probability of the parallel system reaching age t is maximized when assigning component i to location i , for $i=1, \dots, n$.

Proof: This proposition also can be shown easily through an adjacent pairwise interchange argument and proposition 10. ■

3. Bounds on the Distributions of DSR Random Variables

In reliability theory a great amount of research has been done on the development of bounds on distributions with a monotone failure rate. Barlow and Marshall (1964, 1965, 1967) developed bounds for distributions that are IFR, DFR, IFRA, distributions with PF_2 densities and distributions with densities that are descreasing. These bounds were developed based on a certain amount of information on the distributions. The following possibilities were considered with regard to the information that may be available:

- (a) The first moment,
- (b) the first and second moment,
- (c) a percentile and
- (d) conditions on the failure rate.

In this section we explain how the results of Barlow and Marshall can be extended in order to obtain bounds for DSR distributions. We will deal first with nonnegative random variables that have an upper bound, i.e. random variables with a finite support, and subsequently with random variables that have no upper bound. Observe that IFR (DSR) random variables that are truncated from above and (or) from below preserve their IFR-ness (DSR-ness).

Consider a nonnegative random variable S that is bounded from above by U , i.e. $P(S \leq U) = 1$ and has a given mean v . Clearly, when S is DSR, the random variable $T = U - S$ is IFR with mean $\mu = U - v$. Barlow and Marshall (1965) developed lower bounds for the survival probabilities of the random variable T under the assumption that μ is known, i.e.

$$P(T \geq t) \geq \begin{cases} e^{-t/\mu} & \text{for } t \leq \mu \\ 0 & \text{for } t > \mu. \end{cases}$$

Transforming these bounds into bounds on the survival probabilities of the random variable S gives us the following *upper* bound:

$$P(S \geq t) \leq \begin{cases} 1 & \text{for } t < v \\ 1 - e^{-(U-t)/(U-v)} & \text{for } t \geq v \end{cases}$$

Barlow and Marshall (1965) developed in the following way an upper bound on the survival probabilities of T : The equation

$$\mu = t \int_0^1 w^x dx$$

has a solution w_0 if and only if $t \geq \mu$. As w_0 is a function of t we write $w_0(t)$. This solution $w_0(t)$ is unique and

$$P(T \geq t) \leq \begin{cases} 1 & \text{for } t \leq \mu \\ w_0(t) & \text{for } t > \mu \end{cases}$$

Barlow and Marshall (1967) developed tables for the values of $w_0(t)$. The following lower bound: now can be obtained for the random variable S :

$$P(S \geq t) \geq \begin{cases} 1 - w_0(U-t) & \text{for } t < v \\ 0 & \text{for } t \geq v \end{cases}$$

These upper and lower bounds for the random variable S are depicted in Figure 1.

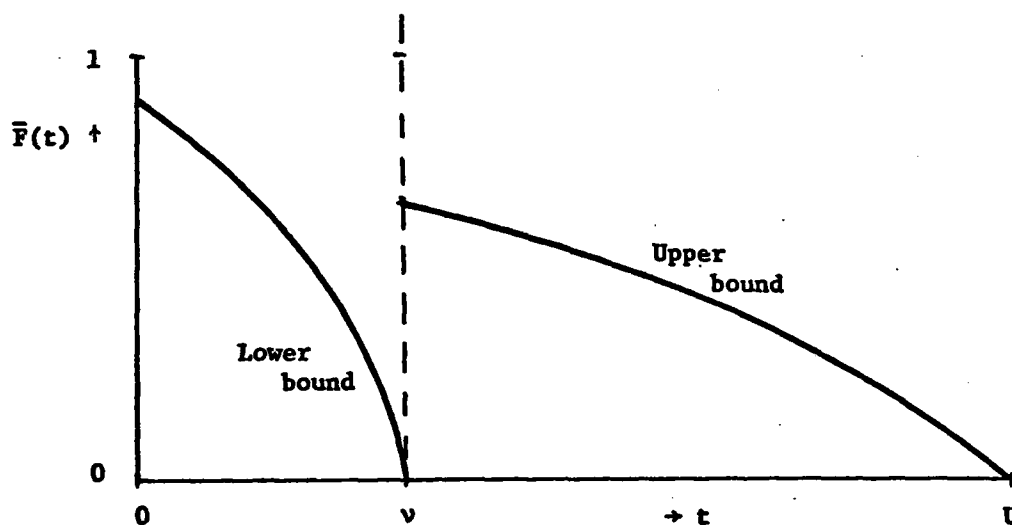


Figure 1

One of the results above can be extended easily for random variables that have no upper bound, provided besides the mean also a percentile ξ_p is known, i.e. $F(\xi_p) = p$, where p is close to 1. Then it can be shown that

$$P(S \geq t) \leq \begin{cases} 1 & \text{for } t \leq v \\ 1 - e^{-(\xi_p - t)/(\xi_p - v)} + (1-p) & \text{for } v \leq t \leq \xi_p \\ p & \text{for } t \geq \xi_p \end{cases}$$

4. Discussion and a Conjecture

It is not easy to get a good intuitive feeling for the survival rate concept. Nevertheless it appears that in many models the survival rates of certain random variables play an important role. It seems useful to know

whether or not random variables have a decreasing survival rate as this implies that the density function has a certain "smoothness".

Moreover in each one of the eight examples presented in Section 1 it appears that the random variables that are DSR also may be IFRA. For example (i) it is known that the lifetime of a parallel system is IFRA, provided the lifetimes of the individual components are IFRA. The same is true for any coherent system, including the k out of n system of example (ii). Ross et al. (1980) have shown for example (iii) that the number of components that are down upon system failure is IFRA for an arbitrary system, provided the lifetimes of the components are i.i.d. For the Poission Shock model of example (iv) it is known that for an arbitrary F , H is IFRA. For the $GI|G|1$ queue of example (v) it can be shown easily that when T_1 is DSR and S_1 is IFRA, D_1 is IFRA. In example (vi), when the processing times of the jobs have IFRA distributions, the distribution of the makespan of the project is IFRA. For the two machine flow shop without intermediate storage of example (vii) the same is true. For the two machine flow shop with infinite intermediate storage of example (viii) the following is true: When the processing times of the jobs on machine 1 are deterministic, not necessarily identical, and the processing times on machine 2 are IFRA, the makespan is IFRA.

We would like to finish this discussion with a conjecture, which illustrates the duality between IFR and DSR random variables once more: Let X_1, X_2, \dots, X_n be i.i.d. IFR (DSR) random variables, then $Y = \phi(X_1, X_2, \dots, X_n)$ is IFR (DSR) provided the function ϕ is increasing concave (convex) in X_i , $i=1, \dots, n$. This conjecture can be easily verified for $n=1$. To get some additional feeling for why this conjecture may be true, it may be useful to

recall that IFR-ness (DSR-ness) is preserved under convolution and that the minimum (maximum) of i.i.d. IFR (DSR) random variables is IFR (DSR); the minimum (maximum) is an increasing concave (convex) function.

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